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LETTER TO THE EDITOR

Binding of a domain wall in the planar Ising ferromagnet

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Abstract. A line of weakened bonds in the interior of a planar Ising ferromagnetic lattice always binds a domain wall. Thus there is no roughening transition in this case, in contrast to the situation with weakened bonds in the surface of a half-planar lattice.

Consider a planar Ising ferromagnet which is characterised by spins $\sigma(i) = \pm 1$ placed at all points (i_1, i_2) of a subset Λ of $\mathbb{Z}^2$, the infinite square lattice with unit side. The energy of a spin configuration $\{\sigma\}$ on Λ is given by

$$E_\Lambda(\{\sigma\}) = -\sum_{|i-j|=1} J(i-j)\sigma(i)\sigma(j) - \sum h(i)\sigma(i) \tag{1}$$

where the $J(k)$ are non-negative couplings and the $h(i)$ are magnetic fields. We shall denote $J((0, 1)) = J_2$, $J((1, 0)) = J_1$.

The probability of the configuration $\{\sigma\}$ is given by

$$p_\Lambda(\{\sigma\}) = Z_\Lambda^{-1} \exp[-\beta E_1(\{\sigma\})] \tag{2}$$

for equilibrium with a heat bath at absolute temperature T with $\beta = 1/k_B T$, k_B being the Boltzmann constant. It will be convenient to use the notation $K_j = \beta J_j$, $j = 1, 2$, hereafter.

It is known that, if $\langle\ \rangle_\Lambda(h, T)$ denotes expectation with respect to (2), then provided $T < T_c$, where T_c solves $\sinh 2K_1 \sinh 2K_2 = 1$, and $H(i) = h$, then (Peierls 1936, Dobrushin 1968, Griffiths 1964, Martin-Löf 1972, Yang 1952, Bennettin *et al* 1973, Abraham and Martin-Löf 1973)

$$\lim_{h\to 0+} \lim_{\Lambda\to\infty} \langle\sigma(0, 0)\rangle_\Lambda(h, T) = m^* \tag{3}$$

where

$$m^* = [1 - (\sinh 2K_1 \sinh 2K_2)^{-2}]^{1/4}. \tag{4}$$

This is, of course, the phenomenon of spontaneous magnetisation. The same limiting result is obtained by taking all $h(i) = 0$, except on the boundary $\partial\Lambda$ where $h(i) = \infty$ and thus only configurations with $\sigma(i) = +1$ on $\partial\Lambda$ are significant in (2). In both cases, $\Lambda \to \infty$ means $(0, 0)$ becomes infinitely far from the boundary. The notion of regulating the state of a system by controlling its periphery *in the infinite volume limit* is perhaps surprising. It is clarified by considering the low-temperature expansion: evidently configurations with neighbouring antiparallel spin pairs are disfavoured. To keep track of such pairs, on the lattice $\Lambda^* = \{i + (\frac{1}{2}, \frac{1}{2}),\ i \in \Lambda \subset \mathbb{Z}^2\}$ draw a unit line segment symmetrically, but perpendicular to the vector separating any antiparallel pair of

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neighbouring spins on Λ. Then, with $\sigma(i)=+1$ on $\partial\Lambda$ there is a 1:1 correspondence between spin and contour configurations, with the proviso that 0, 2 or 4 contour elements meet at any vertex of Λ^*. Let a typical contour configuration on Λ^* have L_x (respectively L_y) contour elements in the (1, 0) (respectively (0, 1)) direction; then the Boltzmann weight is $\exp[-2(K_1L_x+K_2L_y)]$. At low temperatures the contours behave somewhat like a dilute gas. The probability of at least one contour going round the point (0, 0) can be bounded below $\frac{1}{2}$ (Dobrushin 1968, Griffiths 1964, Gallavotti 1972), verifying (3) provided T is small enough.

In order to study the separation of phases, boundary conditions $\mathscr{B}_\Lambda^{+-}$ on the lattice $\Lambda=\{(i_1, i_2): -N\leqslant i_1\leqslant N-1, -M\leqslant i_2\leqslant M-1\}$ are specified so that $\sigma(i)=1$ (respectively -1) for $i\in\partial\Lambda$ whenever $i_2\geqslant 0$ (respectively <0). As $\Lambda\to\infty$, we anticipate a phase of magnetisation $+m^*$ (respectively $-m^*$) far above (respectively below) the line $i_2=0$. From symmetry considerations, the incremental free energy for the associated domain wall should be defined as

$$\tau=-\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{2N+1}\lg[Z(\mathscr{B}_\Lambda^{+-})/Z(\mathscr{B}_\Lambda^{+})] \tag{5}$$

where $\mathscr{B}_\Lambda^{+}$ denotes all boundary spins up.

The profile, or domain wall structure, can be investigated in terms of the function

$$F(y, N)=\lim_{M\to\infty}\langle\sigma(0, y)\rangle_{N,M}^{+-}(0, T) \tag{6}$$

and its limiting behaviour. The following results have been obtained (Abraham and Reed 1974, 1976):

$$\tau=2K_2+\lg\tanh K_1 \tag{7}$$

and

$$\lim_{N\to\infty}F(\alpha N^\delta, N)=\begin{cases}0 & \text{for all } 0\leqslant\delta<\frac{1}{2}\\ m^*\operatorname{sgn}\alpha & \text{for } \delta>\frac{1}{2}\end{cases} \tag{8}$$

with

$$\lim_{N\to\infty}F(\alpha N^{1/2}, N)=m^*\operatorname{sgn}\alpha\ \Phi(b|\alpha|) \tag{9}$$

where

$$b=(\sinh\tau\sinh 2K_1/\sinh 2K_2)^{1/2} \tag{10}$$

and

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp(-u^2)\,\mathrm{d}u. \tag{11}$$

Evidently, the domain wall undergoes large fluctuations. Some insight is obtained by taking the solid-on-solid (sos) limit $K_2\to\infty$ (Temperley 1952); then every vertical line $(n+\frac{1}{2}, y)$ drawn on Λ^* is intersected by one and only one contour element, which therefore belongs to a long contour with ends at $(-N+\frac{1}{2}, -1)$ and $(N+\frac{1}{2}, -\frac{1}{2})$ on Λ. This long contour may be identified uniquely with the domain wall. The probability of shapes of γ is an elementary matter using Markovian methods. Equations (8), (9) and (11) are regained with $m^*=1$, but (10) is replaced by $b_{\mathrm{SOS}}=2\sinh K_2$.

The SOS limit renders the structure of a pure phase trivial. However, there is some evidence that the interface profile has a local structure varying on the scale of the correlation length, the centroid of which fluctuates (Abraham 1981). If we assume that the change from m^* to $-m^*$ is infinitely sharp at the domain wall, then the profile function for the SOS model should be

$$F_{\mathrm{SOS}}(y, N) = m^*[P_N(z<y) - P_N(z \geqslant y)] \tag{12}$$

where z is the unique intercept γ makes with the line $x=0$. This gives (8) and (9) with the correct m^* for a bulk phase. But b_{SOS} only agrees to first order in e^{-2K} with b. Consequently deductions about the domain wall fluctuations based on the SOS model should be made with some caution.

A question of some theoretical importance is the role played by imperfections in reducing domain wall fluctuations. Experimentally, this might correspond to the pinning of domain walls by dislocations, for instance. This Letter gives the exact results for the incremental free energy and boundary profile which obtain when the vertical bond strengths between lines $y=-1$ and $y=0$ are reduced from K_1 to K_0 (in units of $k_B T$). With boundary condition $\mathscr{B}_\Lambda^{+-}$, configurations will be favoured with the largest number of horizontal contour segments lying on the line $y=0$ of Λ^*. This obviously damps fluctuations and must be balanced against the concomitant reduction in entropy. In the SOS limit, with $K_0 = K_1 - \varepsilon$, $K_1 \to \infty$, $\varepsilon > 0$ (fixed) the domain wall always has bounded fluctuations, which diverge as $\varepsilon \to 0$ (Burkhardt 1981, Chalker 1981, Chiu and Weeks 1981, Hilhorst and van Leeuwen 1981). Even though the possibilities for the associated Ising model are considerably more subtle, the same type of result is obtained. Define γ as the unique real solution of

$$\cosh 2K_1^*(\cosh 2K_2\, e^{\gamma} - \cosh 2K_1^*)$$
$$= -e^{-b}[\sinh^2 2K_1^* + e^{\gamma}(\cosh\gamma - \cosh 2K_1^* \cosh 2K_2)] \tag{13}$$

where $\exp 2K_1^* = \coth K$ and

$$e^b = \cosh 2K_1/\cosh 2K_0. \tag{14}$$

Note that (13) and (14) reduce to (21) of Chiu and Weeks (1981) in the SOS limit. Further, notice that $\gamma(b) \sim b^{1/2}$ as $b \to 0$ and that $\gamma(b) = \tau$ when $K_0 = 0$ (recall (7)). The surface tension is given by $\tau = \gamma$, but the profile is

$$F(y, \infty) = \operatorname{sgn} y\{m^*[1-\lambda(y, b)] + \hat{m}\lambda(y, b)\} \tag{15}$$

where $\hat{m}$ is the magnetisation associated with an identical bond perturbation, but $\sigma(i) = +1$ on the boundary. The function $\lambda(y, b)$ is given in terms of a linear Fredholm problem, as is usual in this type of problem (Abraham 1981). Its asymptotic behaviour is $\lambda(y, b) \sim \exp(-\gamma|y|)$ as $y \to \pm\infty$. Thus domain wall binding always occurs for $b \neq 0$.

These results should be compared with the fluctuation damping of a domain wall near the surface of a half-planar lattice. In that case there is a phase transition at a temperature $T_R(K_0)$ ($<T_c$ whenever $K_0 < K_1$) involving unbinding of the domain wall (Abraham 1980). This effect persists in the SOS limit, in its modification to exclude contour height jumps larger than one (Chiu and Weeks 1981) and even in a continuous height model which can be related, through its transfer operator, to a one-particle, one-dimensional Schrödinger problem with an attractive local potential. On the full line, corresponding to the full plane SOS problem, there is always a bound state, but on a half line only if the potential well is deep enough, which corresponds to low enough temperature. Thus the phenomenon described is remarkably robust. Another worthwhile observation is that 'sheet' models give a rather satisfactory qualitative account of the binding–unbinding transition.

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References

Abraham D B 1980 *Phys. Rev. Lett.* **44** 1165
—— 1981 to be published
Abraham D B and Martin-Löf A 1973 *Commun. Math. Phys.* **32** 245
Abraham D B and Reed P 1974 *Phys. Rev. Lett.* **33** 377
—— 1976 *Commun. Math. Phys.* **73** 83
Bennettin G, Gallavotti G, Jona-Lasinio G and Stella A L 1973 *Commun. Math. Phys.* **30** 45
Burkhardt T W 1981 *J. Phys. A: Math. Gen.* **14** L63
Chalker J 1981 to be published
Chiu S T and Weeks J D 1981 *Phys. Rev.* **23B** 2438
Dobrushin R L 1968 *Theory of Probability and its Applications* **13** 197
Gallavotti G 1972 *Riv. Nuovo Cimento* **2** 133
Griffiths R B 1964 *Phys. Rev.* **136A** 437
Hilhorst H J and van Leeuwen J M J 1981 to be published
Martin-Löf A 1972 *Commun. Math. Phys.* **32** 245
Peierls R E 1936 *Proc. Camb. Phil. Soc.* **32** 477
Temperley H N V 1952 *Proc. Camb. Phil. Soc.* **48** 683
Yang C N 1952 *Phys. Rev.* **85** 808